**Unit Overview:** Analytic geometry connects algebra and geometry, resulting in powerful methods of analysis and problem solving. Just as the number line associates numbers with locations in one dimension, a pair of perpendicular axes associates pairs of numbers with locations in two dimensions. This correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof. Geometric transformations of the graphs of equations correspond to algebraic changes in their equations.

Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same way as computer algebra systems allow them to experiment with algebraic phenomena.
Using Algebra to Prove Geometric Properties

LESSON ESSENTIAL QUESTION: How can algebra be used to prove geometric properties?

LEARNING TARGETS:
I can …
• connect any two points on a coordinate grid to a third point so that the three points form a right triangle
• use the right triangle and the Pythagorean Theorem to find the distance between the original two points
• use the slope formula to determine if lines are parallel or perpendicular.
• explain how slope is related to parallel and perpendicular.
• find the coordinates of a point at a specified ratio between two other points.
• calculate the perimeter of any polygon given coordinates.
• calculate the area of triangles and rectangles given coordinates.
• explain the distance formula using Pythagorean Theorem.

CONCEPT OVERVIEW:
Students build on work from 6th grade (finding vertical and horizontal distances on the coordinate plane) to determine the lengths of the legs of the right triangle drawn connecting the points. Students understand that the line segment between the two points is the length of the hypotenuse.

The Pythagorean Theorem will be applied to finding the lengths of segments on a coordinate grid, especially those segments that do not follow the vertical or horizontal lines, as a means of discussing the determination of distances between points. Contextual situations, created by both your students and the teacher, that apply the Pythagorean theorem and its converse should be provided. For example, apply the concept of similarity to determine the height of a tree using the ratio between the student’s height and the length of the student’s shadow. From that, determine the distance from the tip of the tree to the end of its shadow and verify by comparing to the computed distance from the top of the student’s head to the end of the student’s shadow, using the ratio calculated previously. Students will be challenged to identify additional ways that the Pythagorean Theorem is or can be used in real world situations or mathematical problems, such as finding the height of something that is difficult to physically measure, or the diagonal of a prism.

Once your student has had plenty of practice finding distances between two points using the Pythagorean Theorem, they can begin using the distance formula.

They will review the concept of slope as the rate of change of the y-coordinate with respect to the x-coordinate for a point moving along a line, and derive the slope formula. They will use similar triangles to show that every non-vertical line has a constant slope. They should review the point-slope, slope-intercept and standard forms for equations of lines... remembering that lines can be horizontal, vertical, or neither.

Students may use a variety of different methods to construct a parallel or perpendicular line to a given line and calculate the slopes to compare the relationships. They will investigate pairs of lines that are known to be parallel or perpendicular to each other and discover that their slopes are either equal or have a product of −1, respectively. They need to pay special attention to the slope of a line and its applications in analyzing properties of lines.

Students should use slopes and the Euclidean distance formula to solve problems about figures in the coordinate plane such as:
• Given three points, are they vertices of an isosceles, equilateral, or right triangle?
• Given four points, are they vertices of a parallelogram, a rectangle, a rhombus, or a square?
• Given the equation of a circle and a point, does the point lie outside, inside, or on the circle?
• Given the equation of a circle and a point on it, find an equation of the line tangent to the circle at that point.
• Given a line and a point not on it, find an equation of the line through the point that is parallel to the given line.
• Given a line and a point not on it, find an equation of the line through the point that is perpendicular to the given line.
• Given the equations of two non-parallel lines, find their point of intersection.

When given two points, use the distance formula to find the coordinates of the point halfway between them. They will also use linear interpolation to generalize the midpoint formula and find the point that partitions a line segment in any specified ratio. Finally they will use the distance formula to find the length of each side of a polygon whose vertices are known, and compute the perimeter of that figure.
IMPORTANT VOCABULARY:

Leg – Either of the two shorter sides of a right triangle.

Hypotenuse – In a right triangle, the side opposite to the right angle.

Pythagorean Theorem – a theorem that states that in a right triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the legs.

Area – a measurement of the region enclosed by the sides of a polygon. The area is always expressed in squared units.

Coordinate plane – a two-dimensional region determined by a pair of axes and that uses numerical values to represent the location of an object.

Distance formula – the amount of space between two points or things. Distance is always a non-negative number.

Endpoint – the point at an end of a line segment.

Linear equation – an equation whose solution is a straight line.

Perimeter – the sum of the lengths of the sides of a polygon.

Point – slope form – given a point $(x_1, y_1)$ and a slope $m$, and have you plug it into this formula: $y - y_1 = m(x - x_1)$

Polygon - a closed figure formed by three or more line segments.

Slope – intercept form – one way to write the equation of the line. Uses the form $y = mx + b$, where $m$ is the slope and $b$ is the y-intercept.

Triangle – a polygon with three sides.

SAMPLE PROBLEM(s):

1. Use the Pythagorean Theorem to find the distance between the two given points.

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(-2, 4)

(-3, -6)
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Commentary

The Pythagorean theorem is a theorem which says that for every right triangle, the square of the hypotenuse equals the sum of the squares of the two other sides. The distance formula is a method of determining the distance of a line segment between two points. It can be derived using the Pythagorean theorem.

The distance formula follows:

$x$ and $y$ are the $x,y$ coordinates for the first point

$a$ and $b$ are the $x,y$ coordinates for the second point

$d$ is the distance between the two points

The hypotenuse of the triangle will be the distance between the two points. The distance between these points is given by the formula: $d = \sqrt{(x-a)^2 + (y-b)^2}$

Solution

There are 4 units up from the x-axis to (-2, 4) and 6 units down, to (-3, -6) so the length of the leg is 10 units, in the y-direction. There is one unit between (-2, 4) and (-3, -6) in the x direction, so the length of the leg is 1 unit.

$c = d = \sqrt{(x-a)^2 + (y-b)^2} = \sqrt{(10)^2 + (1)^2} = \sqrt{101} = 10.1 \text{ units}$

$|4 - (-6)| = 10 \text{ units (leg)}$

$|3 - (-2)| = 1 \text{ units (leg)}$
2. Prove or disprove that triangle ABC with coordinates A(-1,2), B(1,5), C(-2,7) is an isosceles right triangle.

Commentary
A right triangle with the two legs (and their corresponding angles) equal. An isosceles right triangle therefore has angles of 45°, 45°, and 90°. An isosceles triangle is a triangle with (at least) two equal sides. This property is equivalent to two angles of the triangle being equal. An isosceles triangle therefore has both two equal sides and two equal angles. The name derives from the Greek iso (same) and skelos (leg). Two lines are said to be perpendicular if they meet at a right angle. Perpendicular lines have slopes that are negative reciprocals or the product equals (-1). The slope, \( m \), of a line measures the steepness of the line. Perpendicular lines have slopes that are negative reciprocals or the product equals (-1). The longest side of the triangle is called the "hypotenuse". In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. If we know the lengths of two sides of a right angled triangle, we can find the length of the third side, but it only works on right angled triangles.

Solution
The triangle is an isosceles right triangle because both legs of the triangle have the same measure and the slope of the legs of a triangle form a 90° angle. Use the slope formula to find the slopes of \( BC \) and \( AB \). \( BC \perp AB \) because \( m_{BC} \cdot m_{AB} = (-0.67) \cdot (1.5) = -1 \). Use the Pythagorean Theorem to find the hypotenuses \( BC \) and \( AB \) formed by each triangle. \( \text{measure}_{BC} = \text{measure}_{AB} = 3.6 \text{ cm} = 3.6 \text{ cm} \). Since the two sides are equal, the triangle ABC is also isosceles.

3. Use slope and distance formula to verify the polygon formed by connecting the points (-3, -2), (5, 3), (9, 9), (1, 4) is a parallelogram.

Commentary
In a quadrilateral, if both pairs of opposite sides parallel and congruent, the quadrilateral is a parallelogram. Given the two points \((x_1, y_1)\) and \((x_2, y_2)\), the distance between these points is given by the formula, \( d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \). Parallel lines have equal slopes. The slope formula is calculated by dividing the change in the y values by the change in the x values.

\[
\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1}
\]

Summary
Both pairs of opposite sides are equal and both pairs of opposite sides are parallel. Therefore, the points form a parallelogram.
4. Take a picture or find a picture which includes a polygon. Overlay the picture on a coordinate plane (manually or electronically). Determine the coordinates of the vertices. Classify the polygon. Use the coordinates to justify the classification.

**Commentary**

Closed shapes or figures in a plane with three or more sides are called polygons. Alternatively, a polygon can be defined as a closed planar figure that is the union of a finite number of line segments. In this definition, you consider closed as an undefined term. The term polygon is derived from a Greek word meaning "many-angled." Polygons first fit into two general categories—convex and not convex (sometimes called concave). The endpoints of the sides of polygons are called vertices. When naming a polygon, its vertices are named in consecutive order either clockwise or counterclockwise. Polygons are also classified by how many sides (or angles) they have. The slope, \( m \), of a line measures the steepness of the line. Slope is associated with "rise over run". Rise means how many units you move up or down from point to point. On the graph that would be a change in the y values. Run means how far left or right you move from point to point. On the graph, that would mean a change of x values. The slope formula is calculated by dividing the change in the y values by the change in the x values. \[
\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1}
\]

**Solution**
The polygon is a closed polygon. The polygon has four sides, so it’s a quadrilateral. Parallel lines have the same slopes. A trapezoid is a quadrilateral with two parallel bases. Use the slope formula to find the slope of both sides. The polygon has one pair of parallel sides, so it’s a trapezoid.
5. Find an equation of a line perpendicular to \( y = 3x - 4 \) that passes through \((3, 4)\).

**Commentary**
Perpendicular lines have slopes that are negative reciprocals or the product equals \((-1)\). The points slope formula is \( y - y_1 = m(x - x_1) \) and is used to determine the equation of a line given a slope and a point on the line.

**Solution**
The slope of the given equation is 3. The slope of a line perpendicular to the given equation is \(-\frac{1}{3}\), because

\[
(3) \cdot \left(-\frac{1}{3}\right) = -1.
\]

Substitute the perpendicular slope and the given point into the point-slope formula \( y - 4 = -\frac{1}{3}(x - 3) \) and simplify to find the equation of the line perpendicular to the given equation that passes through \((3, 4)\).

\[
y = -\frac{1}{3}x + \frac{16}{3}
\]

6. Verify that the distance between two parallel lines is constant. Justify your answer.
7. Given \( A(3, 2) \) and \( B(6, 11) \), find the point that divides the line segment \( AB \) two-thirds of the way from \( A \) to \( B \).

**Solution:** The point two-thirds of the way from \( A \) to \( B \) has \( x \)-coordinate two-thirds of the way from 3 to 6 and \( y \) coordinate two-thirds of the way from 2 to 11. So, \((5, 8)\) is the point that is two-thirds from point \( A \) to point \( B \).
8. Find the point that divides the line segment AB in half.

**Commentary**

If the line segments are vertical or horizontal, you may find the midpoint by simply dividing the length of the segment by 2 and counting that value from either of the endpoints. If the line segments are diagonally positioned, more thought must be paid to the solution. When you are finding the coordinates of the midpoint of a segment, you are actually finding the average (mean) of the x-coordinates and the average (mean) of the y-coordinates.

\[ M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \]

**Solution**

9. Calculate the area of triangle ABC with altitude segment CD, given A (-4,-2), B (8,7), C (1,8) and D (4,4).

**Commentary**

The area of a polygon is the number of square units inside that polygon. Area is 2-dimensional like a carpet or an area rug. A triangle is a three-sided polygon. To find the area of a triangle, multiply the base by the height, and then divide by 2. The division by 2 comes from the fact that a parallelogram can be divided into 2 triangles. Since the area of a parallelogram is, \( A = bh \), the area of a triangle must be one-half the area of a parallelogram. Thus, the formula for the area of a triangle is: \( A = \frac{1}{2}bh \). The base and height of a triangle must be perpendicular to each other. The distance formula can be obtained by creating a triangle and using the Pythagorean Theorem to find the length of the hypotenuse. The hypotenuse of the triangle will be the distance between the two points. The subscripts refer to the first and second points; it doesn’t matter which points you call first or second. Given the two points \((x_1, y_1)\) and \((x_2, y_2)\), the distance between these points is given by the formula: \( d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \)

**Solution**

Use the distance formula to find the length of the base of the triangle and the length of the altitude or height. Use the area formula
to find the area of the triangle.

\[ A = \frac{1}{2} b h = \frac{1}{2} (11.82 \text{ cm} \cdot 3.94 \text{ cm}) = 23.31 \text{ cm}^2 \]

\[ d_{CD} = \sqrt{(4-1)^2 + (4-8)^2} = \sqrt{(3)^2 + (-4)^2} = 3.94 \text{ cm} \]

\[ d_{AB} = \sqrt{(8-(-4))^2 + (7-(-2))^2} = \sqrt{(12)^2 + (9)^2} = 11.82 \text{ cm} \]

Virtual Resources: Using Algebra to Prove Geometric Properties